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Derived ritz vector methods for dynamic analysis of bridges

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ABSTRACT

The evaluation of the vibrational response of large complex structural systems subjected to dynamic loads is computationally expensive and time consuming. Recently developed numerical techniques, the Lanczos and Ritz-Wilson methods, based on the Rayleigh-Ritz transformation can reduce the size of the problem and thus the amount of calculations required for the solution. In this research, a number of Ritz algorithms and analysis procedures based on the Lanczos method and the load-dependent derived Ritz algorithms have been implemented in a comprehensive computer program for the static and dynamic analysis of bridge structures. A new cut-off criterion based on the concept of effective mass participation to generate the truncated set of Ritz vectors for multi-directional seismic response analysis has been developed and implemented in the computer program. Numerical examples of the free vibration and seismic response analysis of bridges are presented.

INTRODUCTION

The dynamic behaviour of many engineering structures, especially those that can be severely affected by strong earthquakes, is an important factor for consideration in their design. The evaluation of the dynamic response of large structural systems subjected to dynamic loads is generally very time consuming and computationally expensive. In the modal superposition method, the vibration mode shapes obtained from the eigenproblem of the system are used as the transformation basis to reduce the size of the dynamic problems and then evaluate the dynamic response of the structural systems. In general, the determination of the natural frequencies and displacement shapes of the required vibration modes for the modal superposition analysis of a large complex system is very costly. Recent research has shown that other numerical techniques, such as the Lanczos method using transformation vectors generated in the Krylov space, can be more efficient in calculating the vibrational frequencies and mode shapes of large sparse matrix systems (Wilson et al. 1982, Nour-Omid and Clough 1984, Léger 1988, Ruchala and Lau 1996).

In the present study, the numerical techniques of the Lanczos and Ritz-Wilson methods for the dynamic response analysis of bridge structures have been further developed and implemented in a comprehensive computer program, specifically designed to analyze complex bridge systems. Numerical examples of frequency and time history response analysis of a highway bridge are presented to illustrate the capabilities of the implemented Ritz algorithms, and to investigate their accuracy and efficiency.

RITZ ALGORITHMS

In the derivation of the Lanczos method solution for the eigenvalue problem of a structural system defined by the stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$, a vector $\mathbf{r}$ is selected as the starting initial vector to generate the coordinate sequence of Ritz vectors $\{\mathbf{r}, \mathbf{K}^{-1}\mathbf{M}\mathbf{r}, (\mathbf{K}^{-1}\mathbf{M})^2\mathbf{r}, \ldots, (\mathbf{K}^{-1}\mathbf{M})^m\mathbf{r}\}$, known as the Krylov sequence. The Gram-Schmidt orthogonalization procedure is applied to each generated vector to make it orthogonal to the vectors derived in the previous steps.

It can be shown that the orthogonalization procedure need be applied only to the two preceding vectors, so that the Lanczos vectors can be efficiently generated by the three term recurrence formula written as follows

$$\mathbf{r}_j = \beta_{j+1}\mathbf{q}_{j+1} = \mathbf{K}^{-1}\mathbf{M}\mathbf{q}_j - \alpha_j\mathbf{q}_j - \beta_j\mathbf{q}_{j-1} \tag{1}$$

where

$$\begin{aligned} \alpha_j &= \mathbf{q}_j^T\mathbf{M}\mathbf{K}^{-1}\mathbf{M}\mathbf{q}_j \\ \beta_j &= (\mathbf{r}_{j-1}^T\mathbf{M}\mathbf{r}_{j-1})^{1/2} \end{aligned} \tag{2}$$

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As a result, a set of **M**-orthonormal vectors $\mathbf{Q}_m = \{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m\}$ is constructed which can be used for the Rayleigh-Ritz reduction of the standard equations of motion expressed as follows

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(s)g(t) \tag{3}$$

where the vectors $\ddot{\mathbf{u}}$, $\dot{\mathbf{u}}$ and $\mathbf{u}$ are respectively the acceleration, velocity, and displacement vectors which describe the response of the structure, **M**, **C** and **K** are respectively the square $n \times n$ mass, damping, and stiffness matrices, $\mathbf{f}(s)$ is a spatially distributed load function, unchanged with time, and g(t) is a time-varying scalar amplitude function.

Assuming the damping property of the system in the form of Rayleigh damping which can be expressed as a linear combination of the mass and stiffness matrices, the equations of motion given by Equation 3 can be transformed from the geometric coordinates $\mathbf{u}$ to the generalized Ritz coordinates $\mathbf{z}$ as follows

$$\mathbf{T}_m\ddot{\mathbf{z}} + [a_0\mathbf{T}_m + a_1\mathbf{I}_m]\dot{\mathbf{z}} + \mathbf{z} = \mathbf{Q}_m^T\mathbf{M}\mathbf{K}^{-1}\mathbf{f}(s)g(t) \tag{4}$$

using the transformation

$$\mathbf{u}(t) = \mathbf{Q}_m\mathbf{z}(t) \tag{5}$$

In Equation 4, $\mathbf{T}_m$ is a tridiagonal matrix made up of the coefficients α_j's in the diagonal and β_j's in the sub-diagonal of the matrix, which can be expressed as follows

$$\mathbf{T}_m = \mathbf{Q}_m^T\mathbf{M}\mathbf{K}^{-1}\mathbf{M}\mathbf{Q}_m \tag{6}$$

The reduced tridiagonal equations of motion can be solved directly by step-by-step integration procedures to determine the dynamic response of the system.

Recently, other Ritz type numerical analysis procedures have been developed independent of the Lanczos algorithm for dynamic analysis of structural system based on the physical consideration of the behaviour of the systems. However, it can be shown that many of these numerical procedures may be considered as variations of the original Lanczos algorithm.

When the initial vector in the original Lanczos algorithm is taken as the static displacement vector obtained from the spatially distributed dynamic load applied as static load to the system, the constructed Lanczos coordinates, denoted by ψ_i's, are called the derived Ritz vectors (DRV). The derived Ritz vectors, $\mathbf{\Psi}$, have the important characteristic that the first DRV vector represents a static response to the dynamic load so that there is no need for a static correction in the analysis.

Another numerical algorithm has been developed by Wilson et al. (1982) to generate the Ritz coordinates by considering also the spatial distribution of the applied load in dynamic response analysis. The Ritz-Wilson algorithm has been modified by Leger (1988) with improvement on the numerical stability of the algorithm. The improvement is achieved by using at each step an improved static vector obtained from a one-step inverse iteration procedure. The modified Ritz-Wilson method is also referred to as the LWYD algorithm. It is important to note here that with the introduction of the updated static vectors in the LWYD algorithm, the generated load-dependent Ritz vectors not longer form a Krylov space. Thus, the LWYD method is not a Lanczos type procedure.

REORTHOGONALIZATION SCHEMES

Although in theory orthogonality to all preceding shapes is ensure by orthogonalizing the new shape against the two preceding shapes, the accumulation of round-off errors and the convergence of an eigenvalue will lead to the loss of orthogonality among the Lanczos vectors. In the present study, a number of reorthogonalization schemes have been implemented in the computer program to prevent the loss of orthogonality among the Lanczos vectors, depending on the required level of orthogonality.

Selective Reorthogonalization

In the developed program, orthogonality among Lanczos vectors can be maintained by orthogonalizing the new Lanczos vector $\mathbf{q}_{j+1}$ against the converged eigenvectors. As the Lanczos algorithm proceeds, orthogonality among the Lanczos vectors starts to deteriorate when one of the eigenvectors begins to converge. The growth of the components of the Lanczos vector along a converged eigenvector indicates the return of a banished eigenvector.

The state of orthogonality between a converged eigenvector, $\mathbf{y}_i$, and the current Lanczos vector, $\mathbf{q}_j$, can be measured by the component of $\mathbf{y}_i$ along $\mathbf{q}_j$ using the following recurrence relationship

$$\tau_{j+1} \simeq \frac{(\theta_i - \alpha_j)\tau_j - \beta_j \tau_{j-1}}{\beta_{j+1}} \tag{7}$$

The three term recurrence formula for τ_{j+1} is updated at each step with τ_0 set to zero at the first step. Whenever τ_{j+1} becomes greater than the tolerance, $\tau_{j+1} \geq \sqrt{\epsilon}$, it signals that the component of the corresponding converged eigenvector $\mathbf{y}_i$ has grown too much along $\mathbf{q}_{j+1}$. The vector $\mathbf{y}_i$ is then explicitly deflated out of $\mathbf{q}_{j+1}$. After orthogonalization τ_{j+1} is set back to ϵ.

Partial Reorthogonalization

To alleviate the problem of the loss of orthogonality due to convergence of an eigenvalue, another semi-orthogonalization scheme called the partial reorthogonalization method proposed by Simon (1984) has been implemented in the computer program. In this scheme, the level of orthogonality of the vector $\mathbf{q}_{j+1}$ with the previous Lanczos vectors is monitored by the inner product $\mathbf{w}_j = \mathbf{q}_{j+1}^T \mathbf{M} \mathbf{Q}_j$ using the following recurrence relationship

$$\beta_{j+1}\mathbf{w}_{j+1} = \mathbf{T}_j \mathbf{w}_j - \alpha_j \mathbf{w}_j - \beta_j \mathbf{w}_{j-1} \tag{8}$$

Whenever any element in the vector $\mathbf{w}_{j+1}$ becomes greater than $\sqrt{\epsilon}$, the Lanczos vector $\mathbf{q}_{j+1}$ is orthogonalized against all previous vectors $\mathbf{Q}_j$. To start the recurrence, $\mathbf{w}_0$ is set to zero and $\mathbf{w}_1 = \mathbf{q}_2^T \mathbf{M} \mathbf{q}_1$. The computation of $\mathbf{w}_{j+1}$ involves only the simple update of two vectors. Furthermore, it is not necessary to reorthogonalize in every step, therefore the overall amount of computation for the reorthogonalization process is reduced as compared to the full reorthogonalization scheme.

Combined Reorthogonalization Scheme

Since in the selective reorthogonalization scheme the new Lanczos vector $\mathbf{q}_{j+1}$ is orthogonalized against the converged eigenvectors rather than against all the previous Lanczos vectors, it costs less in comparison to the partial reorthogonalization procedure. To consider this advantage and to account for the two mechanisms involved in the loss of orthogonality, a combined reorthogonalization scheme has also been implemented. In the combined procedure, the condition of the loss of orthogonality is first determined by Equation 8, followed by the evaluation of Equation 7 to determine the cause of the failure. The appropriate reorthogonalization steps are then taken as discussed earlier. Details of implementation can be found in the reference (Ruchala 1997).

CUT-OFF CRITERION

The above Ritz algorithms have been implemented in the computer program NEABS, Nonlinear Earthquake Analysis of Bridge Systems, specially developed for detailed linear and nonlinear dynamic analysis of bridges (IAI-NEABS 1991). To determine the proper number of Ritz vectors that should be included in a dynamic analysis in order to obtain the required accuracy in a truncated system, a new cut-off criterion capable to consider the effect of multi-directional earthquake loads has been developed and implemented in the computer program. The new cut-off criterion is expressed as follows

$$\eta_L = \frac{\sum_{i=1}^{L}(\sum_{j=1}^{M}\sum_{k=1}^{M}\beta_{ji}\beta_{ki})}{\sum_{j=1}^{M}\sum_{k=1}^{M}\mathbf{r}_j^T \mathbf{M} \mathbf{r}_k} \quad \text{and} \quad \beta_{ji} = \mathbf{r}_j^T \mathbf{M} \mathbf{x}_i \tag{9}$$

where $\mathbf{r}_j$ is the influence coefficient vector for base excitation in the j−direction, L is the number of Ritz vectors used in the analysis and M is the number of direction in which the mass of the system is activated by ground motion. The new cut-off criterion is based on the level of mass participation in the dynamic response of the system. It takes into account the mass excited into motion in all directions by the multi-directional ground motions.

NUMERICAL EXAMPLES

Numerical results obtained from the analysis of a five-span composite concrete slab on steel box girder bridge using the developed computer program are presented in this section. The typical cross section and elevation of the bridge are shown in Figure 1. The analysis model, which consists of the bridge superstructure and the piers, has 594 nodes with 778 elements and a total of 3008 degrees-of-freedom. The bridge is assumed to have a 5% damping in the analysis model.

Results for the first 15 natural vibration frequencies are presented in Table 1. When only the computational time of eigensolution is considered, it is 4.6 times more efficient to extract 15 exact vibration modes by the Lanczos method than by the subspace iteration procedure. The computational CPU time used by both the Lanczos and Ritz-Wilson methods to obtain the approximate solutions are very similar. As can be observed from the results shown in Table 1, the converged approximate solutions by the Ritz algorithms do not necessary correspond to the lowest modes. In fact, the converged results are dispersed over the frequency spectrum, especially in the higher frequency results.

The efficiency of the implemented Ritz algorithms is also investigated by an earthquake response analysis of the same bridge. The seismic load case is the three recorded ground motion components of the 1988 Saguenay earthquake in Quebec, Canada. In the present study, the peak horizontal ground acceleration is scaled to a maximum of 0.2g.

The horizontal transverse component of the displacements is plotted in Figure 2. The displacement time history response shows that the results obtained by using 30 mode shapes are in good agreement with the exact solutions. The results also show that even more accourate results can be obtained by the case of using only 10 derived Ritz vectors. The CPU time required by the Derived Ritz algorithm is only 11% of that required by the subspace iteration method, which represent a substantial saving.

The base moment time history responses are plotted in Figures 3. The results show a significant error in the time history response of the internal forces. The discrepancy is still unacceptably high even though 100 mode shapes are used in the analysis. This is consistent with the predictions by the new cut-off criterion η_L. The dynamic mass represented by 100 mode shapes is only 59.9%, whereas the representation of the dynamic mass by 30 DRV vectors is about 83% of the total activated mass. The good agreement of the results presented in Figure 3 confirms the accuracy of the error estimate by η_L.

The computational time, when load-dependent Ritz vectors are applied as the transformation basis, is reduced by 11 to 16 times in comparison to the CPU time of the exact solution.

CONCLUSIONS

This paper presents a number of Ritz algorithms, including the Lanczos, DRV vectors and Ritz-Wilson methods. These analysis procedures and the combined selective and partial reorthogonalization scheme have been implemented in a computer program for dynamic analysis of bridges. The vibrational properties and the dynamic time history response of a five-span composite concrete slab on steel box girder bridge have been analyzed by the computer program to verify and illustrate the capabilities of the implemented algorithms. For the determination of the number of Ritz vectors needed to be included in the multi-directional seismic response analysis, a cut-off criterion based on the concept of effective mass participation has been developed and verify as the efficient and accurate measure.

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Table 1: Vibration frequencies of example bridge

Total number of DOF - 3008

Mode no.	Exact value [Hz]	Converged value Subspace iteration method [Hz]	Converged value Lanczos* method (39 Lanczos vectors) [Hz]	Converged value Lanczos** method (36 DRV vectors) [Hz]	Approx. value Lanczos** method (15 Lanczos vectors) [Hz]	Approx. value Lanczos** method (15 DRV vectors) [Hz]	Approx. value LWYD** method (15 Wilson's Ritz vectors) [Hz]
1	0.962	0.962	0.962	0.962	0.962	0.962	0.962
2	1.368	1.368	1.368	1.368	1.368	1.368	1.368
3	1.896	1.896	1.896	1.896	1.896	1.896	1.896
4	2.095	2.095	2.095	2.095	2.095	2.095	2.095
5	2.424	2.424	2.424	2.424	2.424	2.424	2.424
6	2.804	2.804	2.804	2.804	2.804	2.804	2.804
7	3.063	3.063	3.063	3.063	3.073		
8	3.117	3.117	3.117	3.117	3.120	3.117	3.117
9	3.126	3.126	3.126	3.126		3.126	3.126
10	3.136	3.136	3.136	3.136	3.239		4.023
11	3.969	3.969	3.969	3.969	3.971	3.981	4.024
12	4.087	4.087	4.087	4.087	4.089	4.088	
13	4.530	4.530	4.530	4.530	4.704	4.601	
14	4.689	4.689	4.689	4.689	4.879		4.716
15	4.901	4.901	4.901				
16	5.016					5.183	
17	5.236			5.236	5.503		5.270
18	5.613					5.613	5.618
(20)	6.427				6.678		6.737
(21)	7.117					7.154	
(24)	7.531					7.536	
(28)	8.560						8.757
Total CPU time		86.78 s	36.96 s	36.89 s	33.20 s	31.67 s	31.86 s
CPU time of eigen solution		63.51 s 3.54 s***	13.69 s 3.92 s***	13.62 s 3.85 s***	9.18 s	9.02 s	9.34 s

* - randomly chosen starting vector

** - starting vector chosen as displacement vector obtained from a spatial distribution of load in Y - direction

*** - CPU time of Sturm sequence check included in CPU time of eigensolution

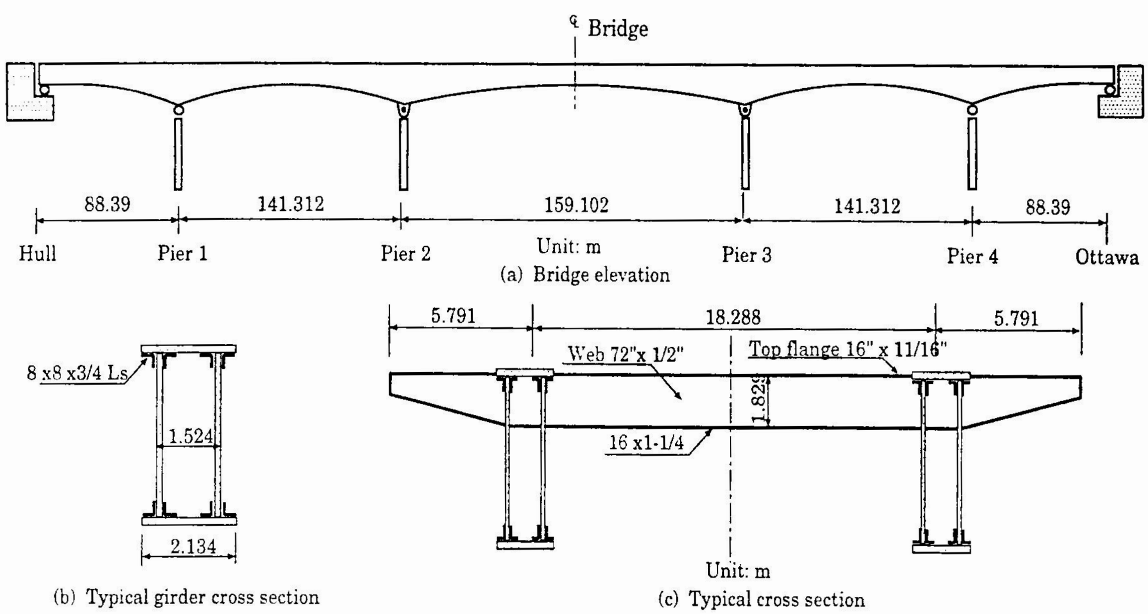
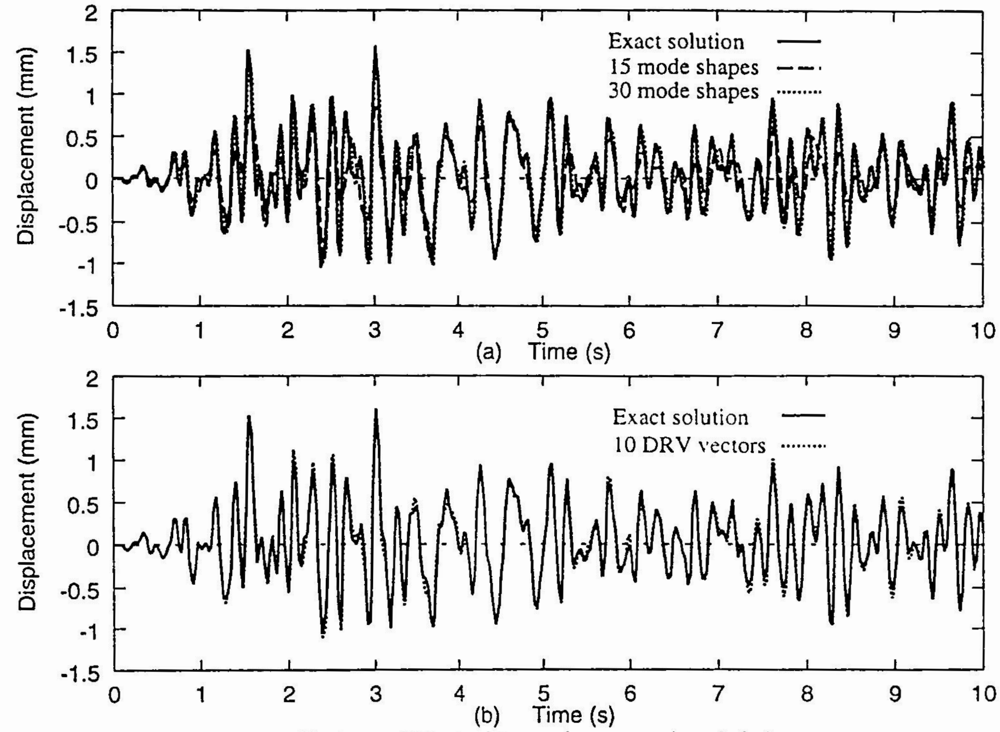
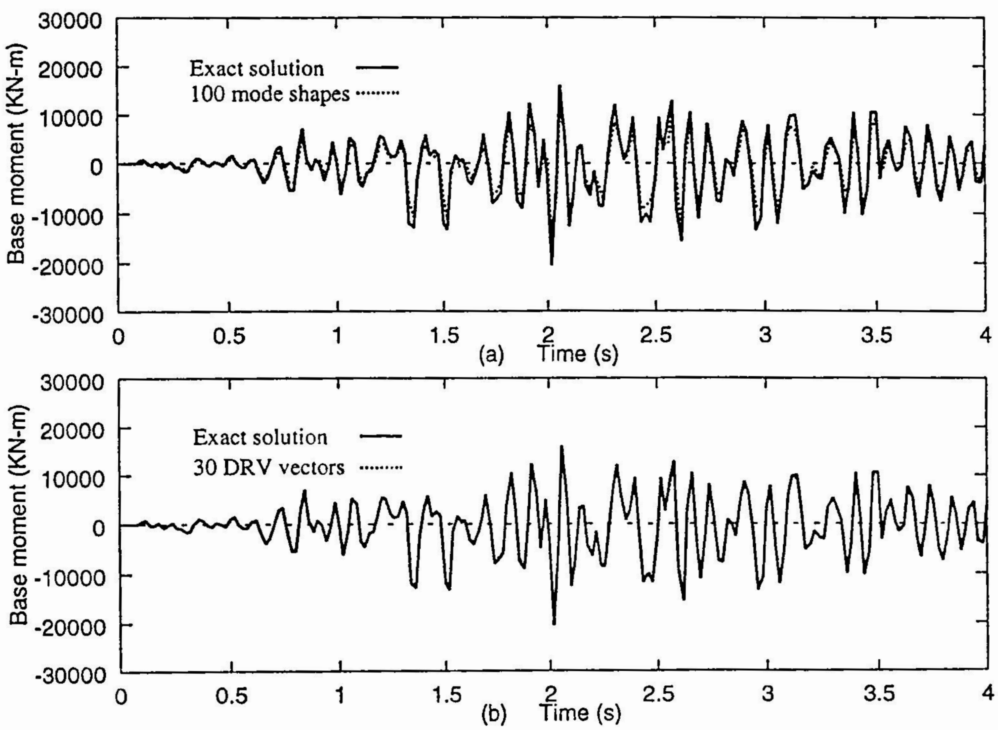


Figure 1: Elevation and cross-sectional dimensions of example bridge



Node no. 222 at mid-span between piers 2 & 3
Transverse translational component

Figure 2: Displacement time history response of example bridge



Transverse component

Figure 3: Base moment time history response of example bridge at Pier 3